## Exercise 6: Object features - moments

### a)

The moments of intertia are the second order central moments, generally defined as

$$u_{pq} = \sum_x \sum_y (x - \overline{x})^p (y - \overline{y})^q f(x, y).$$

This is the moments around the two axes that are parallel to the image axes x and y, passing through the center of mass of the object ($\overline{x}$ and $\overline{y}$).

$$\mu_{20} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (x - \overline{x})^2 f(x, y)$$
$$\mu_{02} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (y - \overline{y})^2 f(x, y)$$
$$\mu_{11} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (x - \overline{x})(y - \overline{y}) f(x, y)$$

### b)

For an 2D object to exhibit a unique orientation the requirement is simply $\mu_{20} \neq \mu_{02}$.

### c)

$$m_{pq} = \sum_x \sum_y x^p y^q f(x, y)$$

First, we need to see that the "same moment of interia about the parallel image coordinate axes (x=0 and y=0)" is

$$m_{20} = \sum_x \sum_y x^2 f(x, y) = (x - 0)^2 f(x, y)$$
$$m_{02} = \sum_x \sum_y y^2 f(x, y) = (y - 0)^2 f(x, y)$$

We also need to know that

$$\overline{x} = \frac{m_{10}}{m_{00}}$$
$$\overline{y} = \frac{m_{01}}{m_{00}}$$

Then we can for example look at the second order moment $\mu_{20}$, and find that:

$$\begin{aligned}
u_{20} &= \sum_x \sum_y (x - \overline{x})^2 f(x, y) \\
&= \sum_x \sum_y (x^2 - 2x\overline{x} + \overline{x}^2) f(x, y) \\
&= m_{20} - 2\overline{x} m_{10} + \overline{x}^2 m_{00} \\
&= m_{20} - 2\overline{x} m_{10} + \overline{x} \frac{m_{10}}{m_{00}} m_{00} \\
&= m_{20} - \overline{x} m_{10}.
\end{aligned}$$

Which is what we wanted to show.

And also for $\mu_{02}$

$$\begin{aligned}
u_{02} &= \sum_x \sum_y (y - \overline{y})^2 f(x, y) \\
&= \sum_x \sum_y (y^2 - 2y\overline{y} + \overline{y}^2) f(x, y) \\
&= m_{02} - 2\overline{y} m_{01} + \overline{y}^2 m_{00} \\
&= m_{02} - 2\overline{y} m_{01} + \overline{y} \frac{m_{01}}{m_{00}} m_{00} \\
&= m_{02} - \overline{y} m_{01}
\end{aligned}$$